

Uniformly continuous orbit equivalence of Markov shifts and gauge actions on Cuntz–Krieger algebras

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Abstract

We introduce a notion of uniformly continuous orbit equivalence as a subequivalence relation of continuous orbit equivalence of one-sided topological Markov shifts. It is described in terms of gauge actions on the associated Cuntz–Krieger algebras and continuous full groups of the Markov shifts.

1 Introduction and Preliminaries

For an $N \times N$ irreducible non permutation matrix $A = [A(i, j)]_{i, j=1}^N$ with entries in $\{0, 1\}$, the shift space X_A of one-sided topological Markov shift (X_A, σ_A) is defined by

$$X_A = \{(x_n)_{n \in \mathbb{N}} \in \{1, \dots, N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\} \quad (1.1)$$

where \mathbb{N} denotes the set of positive integers. It is a compact Hausdorff space by the relative topology of $\{1, \dots, N\}^{\mathbb{N}}$ with the infinite product topology. It has a shift transformation σ_A defined by $\sigma_A((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$. The topological dynamical system (X_A, σ_A) is called the one-sided topological Markov shift defined by the matrix A . The author has introduced a notion of continuous orbit equivalence in the class of one-sided topological Markov shifts to classify the Cuntz–Krieger algebras in [8]. One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be continuously orbit equivalent written $(X_A, \sigma_A) \underset{\text{coe}}{\sim} (X_B, \sigma_B)$ if there exist a homeomorphism $h : X_A \rightarrow X_B$ and continuous functions $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+$, $k_2, l_2 : X_B \rightarrow \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A, \quad (1.2)$$

$$\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)) \quad \text{for } y \in X_B \quad (1.3)$$

where \mathbb{Z}_+ denotes the set of nonnegative integers. The functions $c_1 = l_1 - k_1, c_2 = l_2 - k_2$ are called the cocycle functions for h, h^{-1} , respectively. The continuous orbit equivalence class of (X_A, σ_A) naturally yields a subgroup of homeomorphism group on X_A which is called the continuous full group written Γ_A . The group Γ_A consists of homeomorphisms τ on X_A such that there exist continuous functions $k, l : X_A \rightarrow \mathbb{Z}_+$ such that

$$\sigma_A^{k(x)}(\tau(x)) = \sigma_A^{l(x)}(x) \quad \text{for all } x \in X_A. \quad (1.4)$$

The group Γ_A has been written $[\sigma_A]_c$ in the earlier papers ([8], [9]). The Cuntz–Krieger algebra \mathcal{O}_A is defined by the universal C^* -algebra generated by partial isometries S_1, \dots, S_N satisfying the relations:

$$\sum_{j=1}^N S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^N A(i, j) S_j S_j^*, \quad i = 1, \dots, N. \quad (1.5)$$

We denote by \mathcal{D}_A the C^* -subalgebra of \mathcal{O}_A generated by the projections of the form: $S_{\mu_1} \cdots S_{\mu_n} S_{\mu_n}^* \cdots S_{\mu_1}^*$, $\mu_1, \dots, \mu_n = 1, \dots, N$. The subalgebra \mathcal{D}_A is naturally isomorphic to the commutative C^* -algebra $C(X_A)$ of the complex valued continuous functions on X_A by identifying the projection $S_{\mu_1} \cdots S_{\mu_n} S_{\mu_n}^* \cdots S_{\mu_1}^*$ with the characteristic function $\chi_{U_{\mu_1 \cdots \mu_n}} \in C(X_A)$ of the cylinder set $U_{\mu_1 \cdots \mu_n}$ for the word $\mu_1 \cdots \mu_n$.

H. Matui and the author have finally reached the following classification result:

Theorem 1.1 ([14], cf. [8], [10], [11], [17]). *Let A and B be irreducible, non permutation matrices with entries in $\{0, 1\}$. The following are equivalent.*

- (i) (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.
- (ii) The Cuntz–Krieger algebras \mathcal{O}_A and \mathcal{O}_B are isomorphic and $\det(\text{id} - A) = \det(\text{id} - B)$.
- (iii) There exists an isomorphism $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ such that $\Phi(\mathcal{D}_A) = \mathcal{D}_B$.
- (iv) The continuous full groups Γ_A and Γ_B are isomorphic as groups.
- (v) There exists a homeomorphism $h : X_A \rightarrow X_B$ such that $h \circ \Gamma_A \circ h^{-1} = \Gamma_B$.

For $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$, the correspondence $S_i \rightarrow e^{2\pi\sqrt{-1}t} S_i$, $i = 1, \dots, N$ gives rise to an automorphism of \mathcal{O}_A denoted by ρ_t^A . The automorphisms ρ_t^A , $t \in \mathbb{T}$ yield an action of \mathbb{T} on \mathcal{O}_A called the gauge action. The gauge action is a basic tool to analyze the structure of the Cuntz–Krieger algebra \mathcal{O}_A as in [2] and is closely related to dynamical structure of the underlying topological Markov shift (X_A, σ_A) . Let us denote by \mathcal{F}_A the fixed point subalgebra of \mathcal{O}_A under the gauge action ρ^A . It is well-known that \mathcal{F}_A is an AF algebra whose K_0 -group is known as the dimension group of the underlying topological Markov shift. The subalgebra of \mathcal{F}_A consisting of diagonal elements coincides with the maximal commutative C^* -subalgebra \mathcal{D}_A defined above. There is a discrete subgroup of Γ_A which gives rise to unitaries of finite dimensional subalgebras of the AF algebra \mathcal{F}_A . It is a group of homeomorphism τ in Γ_A for which one may take $k(x) = l(x)$ in (1.4). Since the function $k(x)$ is continuous, it may be chosen to be a constant number written K_τ . We write the group as Γ_A^{AF} , which has been written $[\sigma_A]_{\text{AF}}$ in [8, Section 7]. One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be *uniformly orbit equivalent* ([8, Section 7]) if there exist a homeomorphism $h : X_A \rightarrow X_B$ such that for any $\tau_1 \in \Gamma_A^{\text{AF}}$, $\tau_2 \in \Gamma_B^{\text{AF}}$, there exist continuous functions $k_1 : X_A \rightarrow \mathbb{Z}_+$, $k_2 : X_B \rightarrow \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}(h(\tau_1(x))) = \sigma_B^{k_1(x)}(h(x)) \quad \text{for } x \in X_A, \quad (1.6)$$

$$\sigma_A^{k_2(y)}(h^{-1}(\tau_2(y))) = \sigma_A^{k_2(y)}(h^{-1}(y)) \quad \text{for } y \in X_B. \quad (1.7)$$

This situation is written $(X_A, \sigma_A) \underset{\text{uoe}}{\sim} (X_B, \sigma_B)$. In this case, both functions k_1, k_2 are continuous, so that one may take them to be natural numbers, written K_{τ_1}, K_{τ_2} , such that

$$\sigma_B^{K_{\tau_1}}(h(\tau_1(x))) = \sigma_B^{K_{\tau_1}}(h(x)) \quad \text{for } x \in X_A, \quad (1.8)$$

$$\sigma_A^{K_{\tau_2}}(h^{-1}(\tau_2(y))) = \sigma_A^{K_{\tau_2}}(h^{-1}(y)) \quad \text{for } y \in X_B. \quad (1.9)$$

The following proposition has been seen in [8].

Proposition 1.2 (cf. [8, Theorem 7.4], [5], [6], [19]). *Let A and B be irreducible, non permutation matrices with entries in $\{0, 1\}$. The following are equivalent.*

- (i) (X_A, σ_A) and (X_B, σ_B) are uniformly orbit equivalent.
- (ii) There exists an isomorphism $\Phi : \mathcal{F}_A \rightarrow \mathcal{F}_B$ such that $\Phi(\mathcal{D}_A) = \mathcal{D}_B$.
- (iii) There exists a homeomorphism $h : X_A \rightarrow X_B$ such that $h \circ \Gamma_A^{\text{AF}} \circ h^{-1} = \Gamma_B^{\text{AF}}$.

We note that canonical maximal abelian C^* -subalgebras of \mathcal{F}_A are unique up to isomorphism on \mathcal{F}_A (cf. [19]). Hence the above condition (ii) is equivalent to the condition that \mathcal{F}_A is isomorphic to \mathcal{F}_B .

In this paper, we will study relationships among classification of gauge actions on Cuntz–Krieger algebras, continuous orbit equivalence and continuous full groups. We will show an analogue of Theorem 1.1 for gauge actions referring to Proposition 1.2. In the definition of continuous orbit equivalence, if one may take $l_1(x) = k_1(x) + 1, x \in X_A$ in (1.2) and $l_2(y) = k_2(y) + 1, y \in X_B$ in (1.3), then (X_A, σ_A) and (X_B, σ_B) are said to be *eventually one-sided conjugate* ([13]). This situation is written $(X_A, \sigma_A) \underset{\text{event}}{\approx} (X_B, \sigma_B)$. In this case, one may take the functions k_1, k_2 to be the constant functions taking its values $K_1 = \text{Max}\{k_1(x) \mid x \in X_A\}, K_2 = \text{Max}\{k_2(y) \mid y \in X_B\}$. Hence it is easy to see that $(X_A, \sigma_A) \underset{\text{event}}{\approx} (X_B, \sigma_B)$ if and only if there exist a homeomorphism $h : X_A \rightarrow X_B$ and natural numbers $K_1, K_2 \in \mathbb{N}$ such that

$$\sigma_B^{K_1}(h(\sigma_A(x))) = \sigma_B^{K_1+1}(h(x)) \quad \text{for } x \in X_A, \quad (1.10)$$

$$\sigma_A^{K_2}(h^{-1}(\sigma_B(y))) = \sigma_A^{K_2+1}(h^{-1}(y)) \quad \text{for } y \in X_B. \quad (1.11)$$

For the eventually conjugacy, we have obtained the following result.

Proposition 1.3 ([13, Corollary 3.5]). *Let A and B be irreducible, non permutation matrices with entries in $\{0, 1\}$. (X_A, σ_A) and (X_B, σ_B) are eventually one-sided conjugate if and only if there exists an isomorphism $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ such that*

$$\Phi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \Phi \circ \rho_t^A = \rho_t^B \circ \Phi, \quad t \in \mathbb{T}.$$

We introduce a unified notion of continuous orbit equivalence and uniformly orbit equivalence in the following way.

Definition 1.4. One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be *uniformly continuously orbit equivalent* if there exist a homeomorphism $h : X_A \rightarrow X_B$ and continuous functions $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+, k_2, l_2 : X_B \rightarrow \mathbb{Z}_+$ satisfying (1.2) and (1.3) and for any $\tau_1 \in \Gamma_A^{\text{AF}}, \tau_2 \in \Gamma_B^{\text{AF}}$, there exist natural numbers $K_{\tau_1}, K_{\tau_2} \in \mathbb{N}$ satisfying (1.8), (1.9).

This situation is written $(X_A, \sigma_A) \underset{\text{ucoe}}{\sim} (X_B, \sigma_B)$.

In the presented paper, we will show the following theorem.

Theorem 1.5. *Let A and B be irreducible, non permutation matrices with entries in $\{0, 1\}$. The following are equivalent.*

- (i) (X_A, σ_A) and (X_B, σ_B) are eventually one-sided conjugate.
- (ii) (X_A, σ_A) and (X_B, σ_B) are uniformly continuously orbit equivalent.
- (iii) There exists an isomorphism $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ such that

$$\Phi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \Phi \circ \rho_t^A = \rho_t^B \circ \Phi, \quad t \in \mathbb{T}$$

- (iv) There exists an isomorphism $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ such that

$$\Phi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \Phi(\mathcal{F}_A) = \mathcal{F}_B$$

- (v) There exists an isomorphism $\xi : \Gamma_A \rightarrow \Gamma_B$ of groups such that $\xi(\Gamma_A^{\text{AF}}) = \Gamma_B^{\text{AF}}$.

- (vi) There exists a homeomorphism $h : X_A \rightarrow X_B$ such that

$$h \circ \Gamma_A \circ h^{-1} = \Gamma_B \quad \text{and} \quad h \circ \Gamma_A^{\text{AF}} \circ h^{-1} = \Gamma_B^{\text{AF}}.$$

Before ending this section, we provide several notations and a lemma which will be useful in the proof of Theorem 1.5 in the next section. For $n \in \mathbb{N}$, we denote by $B_n(X_A)$ the set of admissible words of X_A with length n . We denote by $C(X_A, \mathbb{Z})$ the set of integer valued continuous functions on X_A . It has a natural structure of abelian group by pointwise addition of functions. For $f \in C(X_A, \mathbb{Z})$ and $k \in \mathbb{N}$, we set $f^k(x) = \sum_{i=0}^{k-1} f(\sigma_A^i(x))$ for $x \in X_A$. As f is regarded as an element of \mathcal{D}_A , we may define an automorphism $\rho_t^{A,f}$ for $t \in \mathbb{R}/\mathbb{Z}$ on \mathcal{O}_A by setting

$$\rho_t^{A,f}(S_i) = e^{2\pi\sqrt{-1}tf} S_i, \quad i = 1, \dots, N$$

which gives rise to an action of \mathbb{T} on \mathcal{O}_A . If in particular $f \equiv 1_{X_A}$, the action $\rho^{A,f}$ is the gauge action ρ^A .

Suppose that $(X_A, \sigma_A) \underset{\text{coe}}{\sim} (X_B, \sigma_B)$. Let $h : X_A \rightarrow X_B$ be a homeomorphism and $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+, k_2, l_2 : X_B \rightarrow \mathbb{Z}_+$ be continuous functions satisfying (1.2) and (1.3) respectively. As in [15], the homomorphism $\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ defined by

$$\Psi_h(g)(x) = \sum_{i=0}^{l_1(x)-1} g(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)-1} g(\sigma_B^j(h(\sigma_A(x)))) \quad (1.12)$$

for $g \in C(X_B, \mathbb{Z})$, $x \in X_A$ and its inverse $\Psi_{h^{-1}} : C(X_A, \mathbb{Z}) \rightarrow C(X_B, \mathbb{Z})$ gives rise to an isomorphism of groups. By [13], there exists an isomorphism $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ such that

$$\Phi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \Phi \circ \rho_t^{A, \Psi_h(g)} = \rho_t^{B,g} \circ \Phi, \quad g \in C(X_B, \mathbb{Z}), t \in \mathbb{T}.$$

Lemma 1.6. *Let (X_A, σ_A) and (X_B, σ_B) be continuously orbit equivalent given by a homeomorphism $h : X_A \rightarrow X_B$ with continuous functions $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+$, $k_2, l_2 : X_B \rightarrow \mathbb{Z}_+$ satisfying (1.2) and (1.3). Assume that either of the cocycle functions $c_1 = l_1 - k_1$ on X_A or $c_2 = l_2 - k_2$ on X_B is constant. Then both of the functions are 1.*

Proof. Suppose that c_1 is a constant function taking value C_1 . By [14, Lemma 3.3], the equality

$$k_1^{l_2(y)}(h^{-1}(y)) + l_1^{k_2(y)}(h^{-1}(\sigma_B(y))) + 1 = k_1^{k_2(y)}(h^{-1}(\sigma_B(y))) + l_1^{l_2(y)}(h^{-1}(y)) \quad (1.13)$$

holds. Since

$$\begin{aligned} & l_1^{l_2(y)}(h^{-1}(y)) - k_1^{l_2(y)}(h^{-1}(y)) \\ &= \sum_{i=0}^{l_2(y)-1} l_1(\sigma_A^i(h^{-1}(y))) - \sum_{i=0}^{l_2(y)-1} k_1(\sigma_A^i(h^{-1}(y))) = l_2(y)C_1, \end{aligned}$$

and similarly

$$l_1^{k_2(y)}(h^{-1}(\sigma_B(y))) - k_1^{k_2(y)}(h^{-1}(\sigma_B(y))) = k_2(y)C_1,$$

the equality (1.13) ensures us that $l_2(y)C_1 - k_2(y)C_1 = 1$ so that $c_2(y)C_1 = 1$ for all $y \in X_B$. Hence the function c_2 is also constant whose value is written C_2 . We then have $C_1 \cdot C_2 = 1$. Since both C_1 and C_2 are integers, we have $C_1 = C_2 = 1$ or $C_1 = C_2 = -1$. As in [15, Corollary 5.9], we see that

$$\sum_{i=0}^{r-s-1} c_1(\sigma_A^{s+i}(x)) > 0 \quad \text{for } x \in X_A \text{ with } \sigma_A^r(x) = \sigma_A^s(x), r - s > 0.$$

Hence C_1 must be positive, so that we have $C_1 = C_2 = 1$. □

2 Proof of Theorem 1.5

This section is devoting to proving Theorem 1.5.

(i) \implies (ii): Suppose that $(X_A, \sigma_A) \overset{\text{event}}{\approx} (X_B, \sigma_B)$. Take a homeomorphism $h : X_A \rightarrow X_B$ and $K_1, K_2 \in \mathbb{N}$ satisfying (1.10), (1.11). For any $\tau_1 \in \Gamma_A^{\text{AF}}$, there exists K_{τ_1} such that

$$\sigma_A^{K_{\tau_1}}(\tau_1(x)) = \sigma_A^{K_{\tau_1}}(x), \quad x \in X_A.$$

By (1.10), we have

$$\sigma_B^{K_1}(h(\sigma_A^{K_{\tau_1}}(x))) = \sigma_B^{K_1+K_{\tau_1}}(h(x)) \quad \text{for } x \in X_A,$$

so that

$$\sigma_B^{K_1+K_{\tau_1}}(h(\tau_1(x))) = \sigma_B^{K_1}(h(\sigma_A^{K_{\tau_1}}(\tau_1(x)))) = \sigma_B^{K_1+K_{\tau_1}}(h(x)) \quad \text{for } x \in X_A.$$

Similarly there exists $K_{\tau_2} \in \mathbb{N}$ for $\tau_2 \in \Gamma_B^{\text{AF}}$ such that

$$\sigma_A^{K_2+K_{\tau_2}}(h^{-1}(\tau_2(y))) = \sigma_A^{K_2+K_{\tau_2}}(h^{-1}(y)) \quad \text{for } y \in X_B,$$

so that $(X_A, \sigma_A) \underset{\text{ucoe}}{\sim} (X_B, \sigma_B)$.

(ii) \implies (i): A point $x \in X_A$ is said to be eventually periodic if there exist $r, s \in \mathbb{Z}_+$ with $r - s > 0$ such that $\sigma_A^r(x) = \sigma_A^s(x)$. As the matrix A is irreducible and not any permutations, the set X_A^{nep} of non eventually periodic points of X_A is dense in X_A . Suppose that $(X_A, \sigma_A) \underset{\text{ucoe}}{\sim} (X_B, \sigma_B)$. Take a homeomorphism $h : X_A \rightarrow X_B$ and continuous functions $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+, k_2, l_2 : X_B \rightarrow \mathbb{Z}_+$ satisfying the conditions (1.2), (1.3) and (1.8), (1.9). Put $c_i = l_i - k_i, i = 1, 2$. We will first show that both c_1 and c_2 are constant. Suppose that c_1 is not constant. We may find $z \in X_A^{\text{nep}}$ and $\tau \in \Gamma_A^{\text{AF}}$ such that $c_1(z) \neq c_1(\tau(z))$. Since we may take $k \in \mathbb{N}$ such that $\sigma_A^k(z) = \sigma_A^k(\tau(z))$, the set

$$S_0 = \{k \in \mathbb{N} \mid \exists x \in X_A^{\text{nep}}; \exists \tau \in \Gamma_A^{\text{AF}}; c_1(x) \neq c_1(\tau(x)), \sigma_A^k(x) = \sigma_A^k(\tau(x))\}$$

is not empty. We put $K_0 = \text{Min } S_0$. Take $x \in S_0$ and $\tau \in \Gamma_A^{\text{AF}}$ such that

$$c_1(x) \neq c_1(\tau(x)), \quad \sigma_A^{K_0}(x) = \sigma_A^{K_0}(\tau(x)). \quad (2.1)$$

By (1.2) or [15, Lemma 3.1], we have

$$\sigma_B^{k_1^{K_0}(x)}(h(\sigma_A^{K_0}(x))) = \sigma_B^{l_1^{K_0}(x)}(h(x)),$$

so that

$$\sigma_B^{k_1^{K_0}(x)}(h(\sigma_A^{K_0}(\tau(x)))) = \sigma_B^{l_1^{K_0}(x)}(h(x)).$$

Hence we have

$$\sigma_B^{k_1^{K_0}(x) + k_1^{K_0}(\tau(x))}(h(\sigma_A^{K_0}(\tau(x)))) = \sigma_B^{l_1^{K_0}(x) + k_1^{K_0}(\tau(x))}(h(x)),$$

and

$$\sigma_B^{k_1^{K_0}(x) + l_1^{K_0}(\tau(x))}(h(\tau(x))) = \sigma_B^{l_1^{K_0}(x) + k_1^{K_0}(\tau(x))}(h(x)).$$

Since there exists $K \in \mathbb{N}$ such that

$$\sigma_B^K(h(\tau(x))) = \sigma_B^K(h(x)),$$

we have

$$\sigma_B^{k_1^{K_0}(x) + l_1^{K_0}(\tau(x)) + K}(h(\tau(x))) = \sigma_B^{l_1^{K_0}(x) + k_1^{K_0}(\tau(x)) + K}(h(\tau(x))).$$

By the discussion in [15, Section 6], homeomorphism h giving rise to a continuous orbit equivalence preserves eventually periodic points. As $\tau(x) \in X_A^{\text{nep}}$, we see $h(\tau(x)) \in X_A^{\text{nep}}$ so that

$$k_1^{K_0}(x) + l_1^{K_0}(\tau(x)) + K = l_1^{K_0}(x) + k_1^{K_0}(\tau(x)) + K$$

which implies $l_1^{K_0}(x) - k_1^{K_0}(x) = l_1^{K_0}(\tau(x)) - k_1^{K_0}(\tau(x))$ and hence $c_1^{K_0}(x) = c_1^{K_0}(\tau(x))$. This means that

$$\sum_{i=0}^{K_0-1} c_1(\sigma_A^i(x)) = \sum_{i=0}^{K_0-1} c_1(\sigma_A^i(\tau(x))). \quad (2.2)$$

If there exists $m \in \mathbb{N}$ such that $1 \leq m \leq K_0 - 1$ and $c_1(\sigma_A^m(x)) \neq c_1(\sigma_A^m(\tau(x)))$, Put $\bar{x} = \sigma_A^m(x)$. One may find $\bar{\tau} \in \Gamma_A^{\text{AF}}$ such that $\bar{\tau}(\bar{x}) = \sigma_A^m(\tau(x))$. As $c_1(\sigma_A^m(x)) \neq c_1(\sigma_A^m(\tau(x)))$ and $\sigma_A^{K_0}(x) = \sigma_A^{K_0}(\tau(x))$, we have

$$c_1(\bar{x}) \neq c_1(\bar{\tau}(\bar{x})), \quad \sigma_A^{K_0-m}(\bar{x}) = \sigma_A^{K_0-m}(\bar{\tau}(\bar{x})).$$

This is a contradiction of the minimality of K_0 . Hence we see that

$$c_1(\sigma_A^m(x)) = c_1(\sigma_A^m(\tau(x))) \quad \text{for all } m \text{ with } 1 \leq m \leq K_0 - 1. \quad (2.3)$$

By (2.2) and (2.3), we see that $c_1(x) = c_1(\tau(x))$, a contradiction to (2.1). Hence we conclude that c_1 is constant. By Lemma 1.6, we know that $c_1 \equiv c_2 \equiv 1$, so that $(X_A, \sigma_A) \underset{\text{event}}{\approx} (X_B, \sigma_B)$.

(i) \iff (iii): These implications come from Proposition 1.3.

(iii) \implies (iv): This implication is obvious.

(iv) \implies (iii): Assume that there exists an isomorphism $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ satisfying $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Phi(\mathcal{F}_A) = \mathcal{F}_B$. Put $\gamma_t^B = \Phi \circ \rho_t^A \circ \Phi^{-1}$ which is an automorphism for each $t \in \mathbb{T}$. Since $\Phi(\mathcal{F}_A) = \mathcal{F}_B$, we see that $\gamma_t^B(a) = a$ for all $a \in \mathcal{F}_B$. Let us denote by S_1, \dots, S_M the generating partial isometries of \mathcal{O}_B satisfying the relations (1.5) for the matrix B . Put $W_t = \sum_{i=1}^M \gamma_t^B(S_i) S_i^*$. For $\mu = (\mu_1, \dots, \mu_n), \nu = (\nu_1, \dots, \nu_n) \in B_n(X_B)$, we put $S_\mu = S_{\mu_1} \cdots S_{\mu_n}, S_\nu = S_{\nu_1} \cdots S_{\nu_n}$. It then follows that

$$\begin{aligned} W_t S_\mu S_\nu^* &= \sum_{i=1}^M \gamma_t^B(S_i) S_i^* S_{\mu_1} \cdots S_{\mu_n} S_{\nu_n}^* \cdots S_{\nu_1}^* \\ &= \gamma_t^B(S_{\mu_1}) S_{\mu_1}^* S_{\mu_1} \cdots S_{\mu_n} S_{\nu_n}^* \cdots S_{\nu_1}^* \\ &= \gamma_t^B(S_{\mu_1} S_{\mu_1}^* S_{\mu_1} \cdots S_{\mu_n} S_{\nu_n}^* \cdots S_{\nu_1}^* S_{\nu_1}) S_{\nu_1}^* \\ &= \gamma_t^B(S_\mu S_\nu^* S_{\nu_1}) S_{\nu_1}^* \\ &= S_\mu S_\nu^* \gamma_t^B(S_{\nu_1}) S_{\nu_1}^* \\ &= S_\mu S_\nu^* S_{\nu_1} S_{\nu_1}^* \sum_{i=1}^M \gamma_t^B(S_i) S_i^* \\ &= S_\mu S_\nu^* W_t \end{aligned}$$

so that W_t commutes with all elements of \mathcal{F}_B . Since an element of \mathcal{O}_A commuting with \mathcal{F}_A must be scalar, the correspondence $t \in \mathbb{T} \rightarrow W_t \in \mathbb{C}$ gives rise to a character of \mathbb{T} . One may find an integer $c_B \in \mathbb{Z}$ such that $W_t = e^{2\pi\sqrt{-1}c_B t}$, $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$ so that $\gamma_t^B(S_i) = e^{2\pi\sqrt{-1}c_B t} S_i = \rho_t^{B, c_B}(S_i)$, $i = 1, \dots, M$. Hence we have

$$\Phi \circ \rho_t^A = \rho_t^{B, c_B} \circ \Phi, \quad t \in \mathbb{T}. \quad (2.4)$$

Now $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ is an isomorphism such that $\Phi(\mathcal{D}_A) = \mathcal{D}_B$. By [8], there is a homeomorphism $h : X_A \rightarrow X_B$ which gives rise to continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) such that $\Phi(f) = f \circ h^{-1}$ for $f \in C(X_A) = \mathcal{D}_A$. For the homeomorphism $h : X_A \rightarrow X_B$, let $\Phi_h : \mathcal{O}_A \rightarrow \mathcal{O}_B$ be the isomorphism induced from h defined in [8]. It satisfies $\Phi_h(\mathcal{D}_A) = \mathcal{D}_B$ and $\Phi_h = \Phi$ on \mathcal{D}_A . We then have by (1.12) ([13])

$$\Phi_h \circ \rho_t^{A, g} = \rho_t^{B, \Psi_{h^{-1}}(g)} \circ \Phi_h \quad \text{for } g \in C(X_A, \mathbb{Z}).$$

Since $\Psi_{h^{-1}}(1) = l_2 - k_2 = c_2 \in C(X_B, \mathbb{Z})$ is the cocycle function for h^{-1} , we have

$$\Phi_h \circ \rho_t^A = \rho_t^{B, c_2} \circ \Phi_h, \quad t \in \mathbb{T}. \quad (2.5)$$

Put $\alpha = \Psi_h^{-1} \circ \Phi$ which is an automorphism on \mathcal{O}_A such that $\alpha|_{\mathcal{D}_A} = \text{id}$. Let us next denote by S_1, \dots, S_N the generating partial isometries of \mathcal{O}_A satisfying the relations (1.5). By [8, Theorem 6.5 (1)], there exists a unitary one-cocycle $V_\alpha(k)$ in \mathcal{D}_A such that $\alpha(S_\mu) = V_\alpha(k)S_\mu$ for $\mu \in B_k(X_A)$. Hence we have $\alpha(S_i) = V_\alpha(1)S_i, i = 1, \dots, N$ so that

$$\Phi(S_i) = \Phi_h(V_\alpha(1)S_i), \quad i = 1, \dots, N. \quad (2.6)$$

By (2.4) and (2.5), the following equalities hold respectively

$$e^{2\pi\sqrt{-1}t}\Phi(S_i) = \rho_t^{B, c_B}(\Phi(S_i)) = \Phi_h(V_\alpha(1))\rho_t^{B, c_B}(\Phi_h(S_i)) \quad (2.7)$$

and

$$e^{2\pi\sqrt{-1}t}\Phi_h(V_\alpha(1)S_i) = \Phi_h(V_\alpha(1)\rho_t^A(S_i)) = \rho_t^{B, c_2}(\Phi_h(V_\alpha(1))\Phi_h(S_i)) \quad (2.8)$$

$$= \Phi_h(V_\alpha(1))\rho_t^{B, c_2}(\Phi_h(S_i)). \quad (2.9)$$

By (2.6), (2.7) and (2.9), we obtain

$$\rho_t^{B, c_B}(\Phi_h(S_i)) = \rho_t^{B, c_2}(\Phi_h(S_i)), \quad i = 1, \dots, N, t \in \mathbb{T}.$$

Thus we have $c_2 = c_B$ a constant. By Lemma 1.6, we obtain that both cocycle functions c_1 and c_2 are constant 1 so that

$$\Phi \circ \rho_t^A = \rho_t^B \circ \Phi, \quad t \in \mathbb{T}.$$

(ii) \implies (vi): Let $h : X_A \rightarrow X_B$ be a homeomorphism giving rise to a continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) . By [8], one knows that $h \circ \Gamma_A \circ h^{-1} = \Gamma_B$. Since h satisfies (1.8) and (1.9), for any $\tau_1 \in \Gamma_A$, there exists $K_{\tau_1} \in \mathbb{N}$ such that

$$\sigma_B^{K_{\tau_1}}(h \circ \tau_1 \circ h^{-1}(y)) = \sigma_B^{K_{\tau_1}}(y), \quad y \in X_B.$$

Hence we have $h \circ \tau_1 \circ h^{-1} \in \Gamma_B^{\text{AF}}$. We similarly know that $h^{-1} \circ \tau_2 \circ h \in \Gamma_A^{\text{AF}}$ for $\tau_2 \in \Gamma_B^{\text{AF}}$.

(vi) \implies (v): This implication is obvious.

(v) \implies (ii): Suppose that there exists an isomorphism $\xi : \Gamma_A \rightarrow \Gamma_B$ of groups such that $\xi(\Gamma_A^{\text{AF}}) = \Gamma_B^{\text{AF}}$. By the proof of [11, Theorem 7.2], there exists a homeomorphism $h : X_A \rightarrow X_B$ which gives rise to ξ such as

$$\xi(\gamma)(y) = h(\gamma(h^{-1}(y))) \quad \text{for } \gamma \in \Gamma_A, y \in X_B. \quad (2.10)$$

Hence the actions Γ_A on X_A and Γ_B on X_B are topologically conjugate so that $h \circ \Gamma_A \circ h^{-1} = \Gamma_B$. By [8], the homeomorphism $h : X_A \rightarrow X_B$ gives rise to a continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) . By hypothesis, we have

$$h \circ \tau_1 \circ h^{-1} = \xi(\tau_1) \in \Gamma_B^{\text{AF}} \quad \text{for } \tau_1 \in \Gamma_A^{\text{AF}}.$$

Hence for $\tau_1 \in \Gamma_A^{\text{AF}}$, there exists $K'_{\tau_1} \in \mathbb{N}$ such that

$$\sigma_B^{K'_{\tau_1}}(h \circ \tau_1 \circ h^{-1}(y)) = \sigma_B^{K'_{\tau_1}}(y), \quad y \in X_B. \quad (2.11)$$

Put $x = h^{-1}(y) \in X_A$, we have

$$\sigma_B^{K'_1}(h(\tau_1(x))) = \sigma_B^{K'_1}(h(x)), \quad x \in X_A. \quad (2.12)$$

Similarly we have for $\tau_2 \in \Gamma_B^{\text{AF}}$, there exists $K'_{\tau_2} \in \mathbb{N}$ such that

$$\sigma_A^{K'_{\tau_2}}(h^{-1}(\tau_2(y))) = \sigma_A^{K'_{\tau_2}}(h^{-1}(y)), \quad y \in X_B. \quad (2.13)$$

Hence the homeomorphism $h : X_A \rightarrow X_B$ gives rise to a uniformly continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) .

Therefore we complete the proof of Theorem 1.5. \square

3 Subclasses in continuous orbit equivalence class

In this final section, we summarize relationships among several subequivalence relations in continuous orbit equivalence of one-sided topological Markov shifts. Suppose that $(X_A, \sigma_A) \underset{\text{coe}}{\sim} (X_B, \sigma_B)$. Let $h : X_A \rightarrow X_B$ be a homeomorphism and $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+$, $k_2, l_2 : X_B \rightarrow \mathbb{Z}_+$ be continuous functions satisfying (1.2) and (1.3) respectively. As in [15], the homomorphism $\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ defined in (1.12) and its inverse $\Psi_{h^{-1}} : C(X_A, \mathbb{Z}) \rightarrow C(X_B, \mathbb{Z})$ give rise to isomorphisms of the ordered abelian groups between $H^A = C(X_A, \mathbb{Z})/\{g - g \circ \sigma_A \mid g \in C(X_A, \mathbb{Z})\}$ and H^B . By definition of Ψ_h , we have that $\Psi_h(1_{X_B}) = l_1 - k_1 = c_1$ and similarly $\Psi_{h^{-1}}(1_{X_A}) = c_2$. If $[c_1] = [1_{X_A}]$ in H^A and $[c_2] = [1_{X_B}]$ in H^B , (X_A, σ_A) and (X_B, σ_B) are said to be *strongly continuously orbit equivalent*, written $(X_A, \sigma_A) \underset{\text{scoe}}{\sim} (X_B, \sigma_B)$ ([12]). If in particular $c_1 = 1_{X_A}$ and $c_2 = 1_{X_B}$, (X_A, σ_A) and (X_B, σ_B) are eventually one-sided conjugate. Then the following implications hold (cf. [12], [13]).

$$\begin{array}{ccccc}
 & & \text{UOE} & & \\
 & & \uparrow (0) & & \\
 & \text{UCOE} & \xrightarrow{(1)} & \text{SCOE} & \xrightarrow{(2)} \text{COE} \\
 & \Downarrow (3) & & \Downarrow (4) & \\
 \text{one-sided conjugate} & \xrightarrow{(5)} & \text{eventually one-sided conjugate} & & \text{two-sided conjugate}
 \end{array}$$

The implications (0), (1), (2) and (5) are obvious. The implications (3) come from Theorem 1.5. The implication (4) has been shown in [12]. Consider the following matrices

$$\begin{aligned}
 A_2 &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, & F_2 &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, & B_2 &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \\
 B_3 &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, & C_3 &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, & A_4 &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.
 \end{aligned}$$

Since \mathcal{F}_{A_2} and \mathcal{F}_{A_4} are the UHF algebras M_{2^∞} and M_{4^∞} , respectively, there exists an isomorphism $\Phi : \mathcal{F}_{A_2} \rightarrow \mathcal{F}_{A_4}$ such that $\Phi(\mathcal{D}_{A_2}) = \mathcal{D}_{A_4}$, so that $(X_{A_2}, \sigma_{A_2}) \underset{\text{uoe}}{\sim} (X_{A_4}, \sigma_{A_4})$. We however know that $\mathcal{O}_{A_2} = \mathcal{O}_2 \not\cong \mathcal{O}_4 = \mathcal{O}_{A_4}$, so that $(X_{A_2}, \sigma_{A_2}) \not\underset{\text{ucoe}}{\sim} (X_{A_4}, \sigma_{A_4})$. Hence the converse of (0) does not necessarily hold.

By [12], we have shown that $(X_{A_2}, \sigma_{A_2}) \underset{\text{scoe}}{\sim} (X_{B_2}, \sigma_{B_2})$. As we know that

$$K_0(\mathcal{F}_{B_2}) = \mathbb{Z}^3 \xrightarrow{B_2^t} \mathbb{Z}^3 \xrightarrow{B_2^t} \dots,$$

there exists an order preserving isomorphism $\xi : K_0(\mathcal{F}_{B_2}) \rightarrow \mathbb{Z}[\frac{1}{2}](\subset \mathbb{R})$ such that $\xi([1]) = 3 \in \mathbb{R}$. Hence $(K_0(\mathcal{F}_{B_2}), [1]) \not\cong (K_0(\mathcal{F}_{A_2}), [1])$ so that the AF algebra \mathcal{F}_{B_2} is not isomorphic to \mathcal{F}_{A_2} . This shows that $(X_{A_2}, \sigma_{A_2}) \not\underset{\text{ucoe}}{\sim} (X_{B_2}, \sigma_{B_2})$, and the converse of (1) does not necessarily hold.

Since $\mathcal{O}_{A_2} \cong \mathcal{O}_{F_2}$ and $\det(\text{id} - A_2) = \det(\text{id} - F_2)$, we have by Theorem 1.1 $(X_{A_2}, \sigma_{A_2}) \underset{\text{coe}}{\sim} (X_{F_2}, \sigma_{F_2})$, whereas their two-sided topological Markov shifts $(\bar{X}_{A_2}, \bar{\sigma}_{A_2})$ and $(\bar{X}_{F_2}, \bar{\sigma}_{F_2})$ are not topologically conjugate so that $(X_{A_2}, \sigma_{A_2}) \not\underset{\text{scoe}}{\sim} (X_{F_2}, \sigma_{F_2})$. Hence the converse of (2) does not necessarily hold.

Although the two-sided topological Markov shifts $(\bar{X}_{B_3}, \bar{\sigma}_{B_3})$ and $(\bar{X}_{C_3}, \bar{\sigma}_{C_3})$ are topologically conjugate, we know that $\mathcal{O}_{B_3} \cong \mathcal{O}_3 \not\cong \mathcal{O}_3 \otimes M_2(\mathbb{C}) \cong \mathcal{O}_{C_3}$ by [3]. Hence the converse of (4) does not necessarily hold.

The converse of the implication (5) is an open question as in [13].

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